# Math 222A Lecture 6 Notes

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## 1 Quasilinear and Nonlinear First Order PDEs

### 1.1 Quasilinear PDEs and conversation laws

Last time, we were looking at first order, quasilinear, scalar PDEs

$$\sum_j A_j(x,u)\partial_j u + b(x,u) = 0.$$

We saw that our characteristics have to consider both $x$ and $u$. We need to solve the characteristic system

$$\begin{cases} \dot{x} = A(x,u) \\ \dot{u} = -b(x,u) \end{cases}$$

to get a local solution. Because characteristics carry information about $x$ and $u$, there was no prohibition against characteristics intersecting.

What is a noncharacteristic surface in this setting? If our initial data is $u|_\Sigma = u_0$, then the noncharacteristic condition becomes

$$A(x_0, u_0(x_0)) \cdot N \neq 0 \quad \text{on } \Sigma.$$

**Remark 1.1.** The condition of being noncharacteristic depends both on the surface and on the initial data on the surface. So the *problem* is noncharacteristic, rather than the surface (until we have a fixed set of initial data).

The model problem is

$$\begin{cases} \partial_t u + \sum_j A_j(x,u)\partial_j u + b(x,u) = 0 \\ u|_{t=0} = u_0 \end{cases}$$

Since we are already using $t$, let's use $s$ as the parameter along the chracteristics. We have

$$\begin{cases} \dot{t} = 1 \\ \dot{x} = A(x,u) \\ \dot{u} = -b(x,u) \end{cases}$$

The first equation tells us that we can choose $s = t$. This corresponds to a dimensionality reduction of our problem.

**Example 1.1.** A special case of this is what we call **conservation laws**:

$$u_t + \partial_j F_j(u) = 0.$$

We can equivalently write this as

$$u_t + F_j'(u)\partial_j u = 0.$$

Using the first form is not important for scalar equations, but it is for scalar systems because it is not always the case that we can write the second version with a divergence term.

The first version is called **density flux notation**. This is because the $u_t$ tells how the density of some quantity changes in time, and the flux term, $\partial_j F_j(u)$, tells you how the mass is moving with velocity $F_j'(u)$.

## 1.2 Burgers' equation

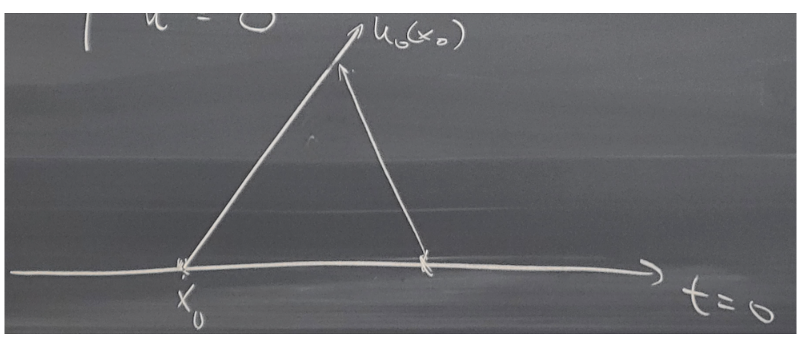
**Example 1.2.** The simplest quasilinear problem is the **Burgers' equation**

$$\begin{cases} u_t + uu_x = 0 \\ u|_{t=0} = u_0. \end{cases}$$

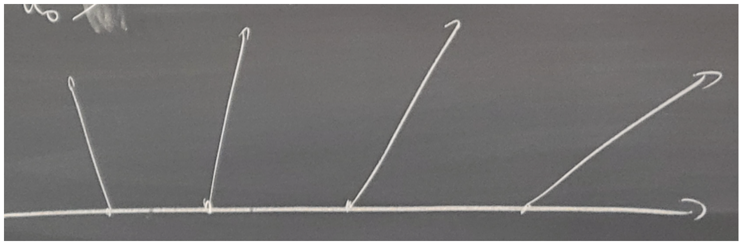
This equation seems simple, but it ends up being a model problem for more complicated equations. Here are the characteristics:

$$\begin{cases} \dot{x} = u \\ \dot{u} = 0. \end{cases}$$

Thus, the characteristics are $x(t) = x_0 + tu_0(x_0)$. Here, the characteristics may intersect as follows:



How would we choose our data so the lines don't intersect? If $u_0$ is increasing, the picture looks like this:



So we get a global solution forward in time, but we don't get a global solutions backward in time. So the only global solutions are constant.

**Remark 1.2.** In physics, we expect there to be *causality*. That is, we expect the future to be determined by the past but not the past to be determined by the future. Later we will see what we will do after the point where characteristics intersect.

Let's give an equation for $u_x$:

$$u_{tx} + uu_{xx} + u_x^2 = 0.$$

If we write $u_x = v$, then this equation is just talking about the derivative along the characteristics:

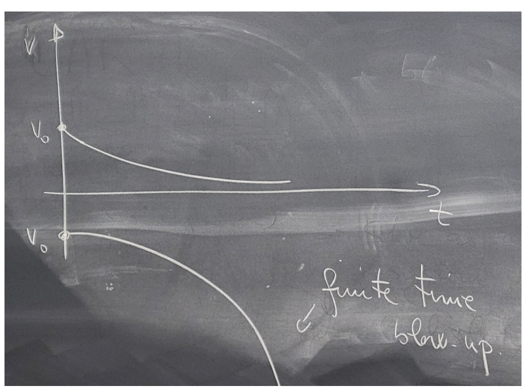
$$(\partial_t + u\partial_x)v + v^2 = 0.$$

We may also write this as

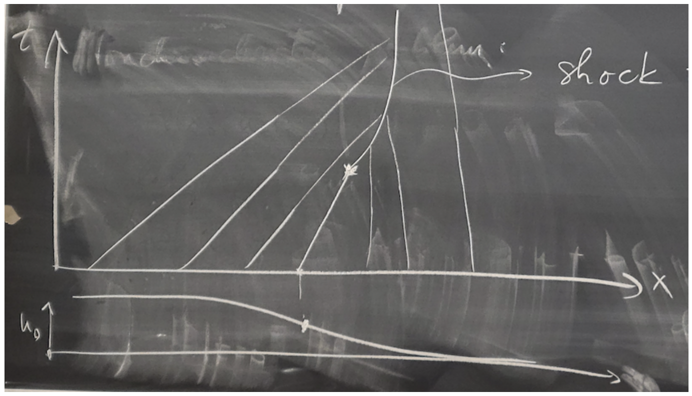
$$\dot{v} + v^2 = 0,$$

where the dot is the derivative along the characteristic. This equation tells us how the slope of the solution is evolving.

If $v_0 > 0$, the slope decreases toward 0. However, if $v_0 < 0$, we get finite time blow-up.



The smallest slope means the fastest blow-up. Suppose the initial data $u_0$ is decreasing, so we will get intersections of characteristics. Then the time with the most negative slope will be the time of blow up for $x$.




Because things intersect, there is no unique way to continue the equation. Here, we have a **shock**, or a jump discontinuity. We will see later how to find what the equation for the shock curve looks like.

Conservation laws is still a very active area, with a number of hard problems.

### 1.3 Fully nonlinear problems

We now look at PDEs of the form

$$\begin{cases} F(x, u, \partial u) = 0 \\ u|_\Sigma = u_0, \end{cases}$$

where the dependence on $\partial u$ is nonlinear. Where do we start? Before, we had a vector field that let us interpret the equation using a directional derivative.

Let's look at the linearized equation: Suppose we have not just a solution but a 1-parameter family of solutions $u^h$ to our problem with solution $v^h$ to our linearized equation given by

$$\frac{d}{dh} u^h = v^h.$$

Differentiate the equation with respect to $h$ to get the linearized equation:

$$0 = \frac{\partial}{\partial h} F(x, u^h, \partial u^h) = F_u \cdot v^h + F_{p_j} \cdot \partial_j v^h,$$

where we write $F = F(x, u, p)$ and $p = (p_1, \ldots, p_n)$ (in $\mathbb{R}^n$).

This linearized equation is a linear transport equation. So we get a vector field $A_j = F_{p_j}(x, u, \partial u)$. We should try to use this vector field to find characteristics. Our equation looks like

$$\begin{cases} \dot{x}_j = F_{p_j}(x, u, \partial u) \\ \dot{u} = \cdots \end{cases}$$

The first equation depends on $\partial u$, so we may try to add an equation $\dot{\partial u} = \cdots$. But then we would get $\partial^2 u$ in this equation, and we would be in the same situation. How do we get past this issue?

Suppose $F(u, \partial u) = 0$. We say that this equation is invariant with respect to translations. This means that if $u(x)$ is a solution, $u(x + hy)$ is a solution, as well. This produces a 1-parameter family of solutions. This implies that $y \cdot \partial u$ solves the linearized equation. In particular, we can use this as our equation for $\dot{\partial u}$. Here is the computation:

$$\begin{cases} \dot{x}_j = F_{p_j}(x, u, \partial u) \\ \dot{u} = F_{p_j}(x, u, \partial u) \cdot \partial u \qquad = \text{directional derivative of } u \text{ in the } F_{p_j} \text{ direction} \\ \dot{\partial_j u} = -F_{x_j}(x, u, \partial u) - F_u(x, u, \partial u) \cdot \partial_j u, \end{cases}$$

where we calculate the last equation by

$$\begin{aligned} 0 &= \partial_j F(x, u, \partial u) \\ &= F_{x_j}(x, u, \partial u) + F_u(x, u, \partial u)\partial_j u + \underbrace{F_{p_k}(x, u, \partial u)\partial_k \partial_j u}_{\dot{\partial_j u}}. \end{aligned}$$

We still have a problem. Suppose we solve the above system. We are treating the function $u$ and its derivatives as separate objects, so how do we know that the solutions are still related to each other? First, let's summarize what we have done so far in a proposition:

**Proposition 1.1.** *If $u \in C^2$, then $(x, u, \partial_j u)$ solve the characteristic system*

$$\begin{cases} \dot{x}_j = F_{p_j}(x, u, \partial u) \\ \dot{u} = F_{p_j}(x, u, \partial u) \cdot \partial u \\ \dot{\partial_j u} = -F_{x_j}(x, u, \partial u) - F_u(x, u, \partial u) \cdot \partial_j u. \end{cases}$$

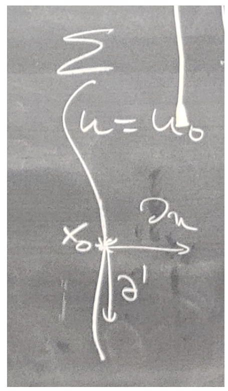
When we solve the system, use the notation $z$ instead of $u$ and $p_j$ instead of $\partial_j u$ because we are solving this equation without enforcing the relationship between these objects. The characteristic system becomes

$$\begin{cases} \dot{x}_j = F_{p_j}(x, z, p) \\ \dot{z} = F_{p_j}(x, z, p) \cdot p_j \\ \dot{p}_j = -F_{x_j}(x, z, p) - F_z(x, z, p) \cdot p_j. \end{cases}$$

What is the initial data for this system? We had $x(0) = x_0$ and $u(0) = u_0$ before, but now we have

$$\begin{cases} x(0) = x_0 \\ u(0) = u_0 \\ \partial u(0) = ? \end{cases}$$

We need the information of *all* the derivatives of $u$ at $x_0$. In particular, we need both $n-1$ tangential derivatives to $\Sigma$ and 1 normal partial derivative to $\Sigma$.




If we frame this in the tangent space, we want the tangent derivative $\partial' = (\partial_1, \ldots, \partial_{n-1})$ and the normal derivative $\partial_n$. We know $\partial'$, but what about $\partial_n$? We know that

$$F(x_0, u_0, \partial' u_0, \partial_n u) = 0,$$

so we would like to solve for $\partial_n u$. This tells us that

$$\partial_n u = G(x_0, u_0, \partial' u_0)$$

for some function $G$. We can do this if

$$F_{p_n}(x_0, u_0, \partial' u_0, p_n) \neq 0.$$

If we did not put our equation in this special frame, this condition reads as

$$F_p(x_0, u_0, p) \cdot N \neq 0,$$

the condition that the equation is noncharacteristic.

**Remark 1.3.** What if this equation has more than 1 solution? We may not get uniqueness; the answer may depend on our choice here of initial data.